Intersection Theory #1

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We consider the subvarieties on a variety. To be specific, suppose we have an n-dimensional variety X. We define $Z_k X$ to be the free Abelian group

$$Z_k(X) = \bigoplus_{V \subset X \ k\text{-dim subvariety}} \mathbb{Z} \cdot V$$

We will use some comparison of intersections in differential topology and in algebraic geometry.

In differential topology, two embedded submanifolds $Y, Z \subset X$ intersect transversely if for every point $p \in Y, Z$ we have $T_p Y + T_p Z = T_p X$. In this case, the intersection of Y and Z is also an embedded submanifold satisfying

$$\text{codim} Y \cap Z = \text{codim} Y + \text{codim} Z$$

For a smooth variety X, we know that $\dim_{\kappa(p)} T_p X = \dim X$, where T_p means $\text{Hom}_{\kappa(p)}(\mathfrak{m}_p/\mathfrak{m}_p^2, \kappa(p))$. Suppose we have two smooth subvarieties $Y, Z \subset X$. They are transverse, or intersect properly, if we have $T_p Y + T_p Z = T_p X$ for every $p \in Y \cap Z$ set-theoretically. If the condition holds only for a dense open subset of $Y \cap Z$ then we say Y, Z intersect generically transversely. The scheme-theoretical intersection of Y and Z is defined by the ideal sheaf $\mathcal{I}_Y + \mathcal{I}_Z$, where $\mathcal{I}_Y$ and $\mathcal{I}_Z$ are ideal sheaves defining Y and Z respectively. In this case, we know that $Y \cap Z$ is a closed subscheme of X with codimenison $\text{codim} Y + \text{codim} Z$.

But what will happen for those non-transverse intersection? For example, what is the self-intersection? In differential topology, we always can use homotopy to make two submanifolds intersect transversely. We also need to define a kind of equivalence(rational equivalence) in algebraic geometry satisfying:

1. Given two subvariety Y, Z, we can find some Y' equivalent to Y and Z' equivalent to Z such that Y' intersect transversely with Z'.

2. For any such choices Y', Z' and Y'', Z'', the intersection $Y' \cap Z'$ should be equivalent to $Y'' \cap Z''$.

If subvarieties above exist then we can define the "intersection class" of Y and Z to be $Y' \cap Z'$. In general, the existence does not hold. But for smooth quasi-projective varieties, the answer is positive. To be specific, we define the group of k-cycles $Z_k(X)$ on X as

$$Z_k(X) = \bigoplus \mathbb{Z} \cdot V \quad V \text{ is a } k\text{-dimensional subvariety of } X$$

And let $A_k(X) = Z_k(X)/\text{Rational equivalence}$.(We will define rational equivalence later) We call that $A(X) = \bigoplus_k A_k(X)$ the Chow ring of X. The well-known moving lemma can be stated as follows:

Theorem 0.1 (Moving Lemma)**.** *Let X be a smooth quasi-projective variety. For every $\alpha, \beta \in A(X)$ there are $A, B \in Z(X) = \bigoplus Z_k(X)$ generically transverse such that $[A] = \alpha$ and $[B] = \beta$. The class $[A \cap B]$ is independent of the choice of A and B.*

Thus we can give following structure theorem on Chow ring of smooth quasi-projective space:

Theorem 0.2. *Let X be a smooth quasi-projective variety. There exists a unique product structure on $A(X)$ such that*

$$\alpha \cdot \beta = [A \cap B] \quad [A] = \alpha,\ [B] = \beta \text{ and } A, B \text{ are generically transverse}$$

The structure makes $A(X)$ into an associative commutative ring graded by codimension.

1 Rational Equivalence

Definition 1.1. *Let X be a scheme. Then we define $Z_k(X)$ to be the free abelian group generated by the set $\{V|V$ is a k-dimensional integral closed subscheme of X. A k-cycle of X is an element of $Z_k(X)$. An effective cycle is $\alpha \in Z_k(X)$ such that for every k-dimensional integral closed subscheme $V \subset X$, the coefficient of V in α is non-negative.*

Before the formal discussion, we take following convention: We always assume all the schemes to be separated. We always assume there is a given base field k.(You can assume it is algebraic closed or exactly $\mathbb{C}$ if you want, which will not cause any loss for most contents. But if some conclusions only holds for 0-character fields or algebraic closed fields, we will point out.) Sometimes we only concern varieties. But it is often the case, that we consider more general schemes of finite type over k. Thus we take following terminologies:

1. "Let X be a scheme" means "X is a separated scheme of finite type over k.
2. "Let X be a variety" means "X is an integral separated scheme of finite type over k".
3. "Let Z be a subvariety of X" means "Z is an integral closed subscheme of X".

1.1 Cycle associated to closed subschemes

In order to define rational equivalence, we need to study some special cycles. We know that $Z_k(X)$ is the free abelian group consists generated by subvarieties. But we usually study more general closed subscheme. It might be non-reduced and non-irreducible. For example, $k[x]/(x)$ and $k[x]/(x^2)$ obviously have the same underlying subset. But obviously they are different in multiplicity. We cannot regard them as the same object in $Z(X)$. So firstly, we need to define the cycle associated to a closed subscheme.

Let X be a scheme. Z be a closed subscheme of X and Z' be an irreducible component of Z with generic point η. Then

$$\mathrm{length}_{\mathcal{O}_{X,\xi}}\mathcal{O}_{Z,\xi} < \infty$$

Since η is a generic point of some irreducible component of Z. We have $\dim(\mathcal{O}_{Z,\eta}) = 0$. Thus it is an Artinian ring. Hence it is of finite length of itself. Since $\mathcal{O}_{Z,\eta}$ is the quotient of $\mathcal{O}_{X,\eta}$, we know the conclusion holds.

Definition 1.2. *Let X be a scheme. And Z be a closed subscheme of X. Let Z' be an irreducible component of Z with generic pt η(We regard Z' as a subvariety of X by its natural reduced closed subscheme structure), the geometric multiplicity of Z' in Z is*

$$m_{Z',Z} = \mathit{length}_{\mathcal{O}_{X,\xi}}\mathcal{O}_{Z,\xi}$$

Suppose Z is of dimension k. Then the k-cycle(or the cycle) associated to Z is

$$[Z]_k = \sum m_{Z',Z}[Z']$$

The sum is over the irreducible components of Z with dimension k. Since Z only has finitely many irreducible components, the sum is also finite.

Remark. *Z' may not be of pure dimension k. But $[Z']$ only concerns its irreducible components of maximal dimension.*

Here the class $[X]$ associated to the scheme X itself is called the fundamental class.

In fact, the definition above can be generalized. Let $\mathcal{F}$ be a coherent sheaf on X. We can define the cycle associated to $\mathcal{F}$. Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. And Z' be an irreducible component of the support of $\mathcal{F}$. Let η be the generic pt of Z'. Then

$$\mathrm{length}_{\mathcal{O}_{X,\xi}}\mathcal{F}_\xi < \infty$$

is finite.

Definition 1.3. *Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. Let Z' be an irreducible component of the support of $\mathcal{F}$ with generic pt η(We regard Z' as a subvariety of X by its natural reduced closed subscheme structure), the multiplicity of Z' in Z is*

$$m_{Z',\mathcal{F}} = length_{\mathcal{O}_{X,\xi}} \mathcal{F}_{Z,\xi}$$

Suppose the support of $\mathcal{F}$ is of dimension k. Then the k-cycle(or the cycle) associated to $\mathcal{F}$ is

$$[\mathcal{F}] = [\mathcal{F}]_k = \sum m_{Z',\mathcal{F}}[Z']$$

The sum is over the irreducible components of the support of $\mathcal{F}$ with dimension k.

In this opinion, we know $[Z]_k$ is the cycle associated to $\mathcal{O}_Z$. It is often the case in the intersection theory that we reduce the proof from cycles to subvarieties. And we can draw the conclusion by the functoriality of coherent sheaves. Furthermore, we have following proposition that relate $Z_k(X)$ and K_0-groups.

Consider following K-group: $K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X))$, the free group generated by $\mathcal{F}$ with support no more than k-dimensional modulo following relation:

1. $[\mathcal{F}] - [\mathcal{F}'] - [\mathcal{F}'']$ if we have short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$.
2. $[\mathcal{G}]$ if $\dim(\mathrm{Supp}(\mathcal{G})) < k$.

Proposition 1.1. *Let X be a scheme. The map*

$$Z_k(X) \longrightarrow K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)), \quad \sum n_Z[Z] \mapsto \left[\bigoplus\nolimits_{n_Z>0} \mathcal{O}_Z^{\oplus n_Z}\right] - \left[\bigoplus\nolimits_{n_Z<0} \mathcal{O}_Z^{\oplus -n_Z}\right]$$

and

$$K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)) \longrightarrow Z_k(X), \quad \mathcal{F} \longmapsto [\mathcal{F}]_k$$

are mutually inverse isomorphisms.

We will return to the relationship between intersection theory and K-theory in the future.

1.2 Principal Divisors

As an analogue of poles and divisors in complex meromorphic functions. We define the cycle associated to a rational function on a variety X. The local ring $\mathcal{O}_{X,Z}$ of the generic point of a subvariety Z with codimension 1 is a noetherian local ring of dimension 1. If the local ring is regular, then it is a discrete valuation ring and we can define $\mathrm{ord}_Z(f)$ to be the valuation of f for every rational function $f \in R(X)^*$. But the conception of order of a rational function along some 1-codimension subvariety can be generalized.

Also, we need a lemma before our definition:

Lemma 1.1. *Let A be a noetherian local ring of dimension 1. Let g be a non-zerodivisor of A. Then $length_A A/gA$ is finite.*

g is not contained in any minimal prime of A. Thus A/gA is a dimension 0 local ring.

Definition 1.4. *Let X be an variety of dimension n and $f \in R(X)^*$ be a rational function. Let Z be an $n-1$ dimensional subvariety of X. The order of vanishing of f along Z is the integer*

$$ord_Z(f) = ord_{\mathcal{O}_{X,\xi}}(f) = length_{\mathcal{O}_{X,\eta}} \mathcal{O}_{X,\eta}/(f)$$

The cycle associated to f is

$$div(f) = div_X(f) = \sum ord_Z(f)[Z]$$

where the sum is over all the $n-1$-dimensional subvarieties of X.

We usually call it the principal Weil divisor of f. The sum is finite by the knowledge of divisors. And we have obvious equation

$$\mathrm{div}_X(fg) = \mathrm{div}_X(f) + \mathrm{div}_X(g)$$

by the additivity of length.

1.3 Rational Equivalence

Just like we define divisors classes. We generalize the "linear equivalence" to cycles of all dimensions.

Definition 1.5. *Let X be a scheme. Let α, β be k-cycles on X. If there exists finitely many $k+1$-subvarieties W_i and rational functions $f_i \in R(W_i)$ such that*

$$\alpha - \beta = \sum div(f_i)$$

then we say that α and β are rational equivalent.

There are various notions for rational equivalent cycles like $\alpha \sim \beta$ or $\alpha \sim_{rat} \beta$. Note that an k-cycles can only be rational equivalent to k-cycles. However, we some times consider α, β that have cycles of different dimensions. Then we say α, β are rational equivalent if $[\alpha]_k = [\beta]_k$ for all k.

Definition 1.6. *Let X be a scheme. The Chow group of k-cycles on X is $A_k(X) = Z_k(X)/\sim_{rat}$. The Chow ring of X is*

$$A(X) = \bigoplus A_k(X)$$

Up to now, we haven't give the multiplicative structure on the Chow ring. So we only regard it as a graded abelian group. Some materials also use $A^k(X)$ to denote the Chow group of codimension k-cycles. Obviously, if $\dim X = n$. Then $A^n(X)$ is the free group generated by its irreducible components of maximal degree since rational equivalence is inherent from higher dimensional varieties.

The definition above is straightforward but a little lack of geometric prospect. Let us consider $X \times_k \mathbb{P}^1_k \simeq X \times \mathbb{P}^1$. Suppose Z is a subvariety of X and f is a rational function on Z. We know that f can be regarded as a map from Z to $\mathbb{P}^1$. Now consider the "graph" Z' of f, which is a subset of $X \times \mathbb{P}^1$. It is isomorphic to X since the natural projection induce isomorphism. Then the intersection of Z' with $\{t = 0\}$ is exactly the zeros of the rational function. And its intersection with $\{t = \infty\}$ is exactly the poles of the rational function. Projective the zeros and the poles to X and then we get two rational equivalent cycles.

Here $\{t = 0\}$ means the subset $X \times D_0$ of $X \times \mathbb{P}^1$, where D_0 is the point corresponding to zero. And $\{t = \infty\}$ means the subset $X \times D_\infty$, where D_∞ is the infinite point of the projective line.

Lemma 1.2 (Graph of a Rational Function)**.** *Let X be an n-dimensional variety and $f \in R(X)^*$. Let $U \subset X$ be a nonempty open such that f corresponding to a section $f \in \Gamma(U, \mathcal{O}_X^*)$. Let $Y \subset X \times \mathbb{P}^1$ be the closure of the graph of $f : U \to \mathbb{P}^1$. Then*

1. *The projection $p : Y \to X$ is proper.*
2. *$p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is an isomorphism.*
3. *$Y_0 = Y \cap \{t = 0\}$ and $Y_\infty = Y \cap \{t = \infty\}$ are $n-1$-dimensional effective Cartier divisors of Y.*
4. *We have $div_Y(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$. And $div_X(f)$ is the image of $div_Y(f)$ under the projection.*
5. *If we view Y_0 and Y_∞ as closed subschemes of X, then we have*

$$div_X(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$$

We can generalize the conclusion. In fact, we can consider subvarieties of $X \times \mathbb{P}^1$ not contained in $\{t = 0\}$ and $\{t = \infty\}$,

Proposition 1.2. *Let X be a scheme. Let $W \subset X \times \mathbb{P}^1$ be an subvariety of dimension $k+1$. Suppose $W \not\subset \{t = 0\}$ and $W \not\subset \{t = \infty\}$. Let $W_0 = W \cap \{t = 0\}$ and $W_\infty = W \cap \{t = \infty\}$. Then:*

1. *W_0, W_∞ are effective Cartier divsors of W, i.e., k-dimensional subvarieties of $X \times \mathbb{P}^1$.*
2. *W_0, W_∞ can be viewed as closed subschemes of X via the projection map $X \times \mathbb{P}^1 \to X$ and $[W_0]_k \sim [W_\infty]_k$.*
3. *Suppose $\alpha \in Z_k(X)$ is rational equivalent to 0. Then we can find finitely many $k+1$ dimensional subvarities W_i of $X \times \mathbb{P}^1$ such that*

$$\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$$

Proof. 1. We know that $\mathbb{A}^1$ is an open subset of $\mathbb{P}^1$ that contains the zero point D_0. Thus we consider $W' = W \cap X \times \mathbb{A}^1$ since $W' \cap \{t = 0\} = W \cap \{t = 0\}$. Since we know $W \neq W_\infty$. W' is still an $k+1$-dimensional variety. And t defines a regular section on W' that not identically zero. Then $W' \cap \{t = 0\}$ is exactly the effective Cartier divisor of W that determined by (t). By similar argument, we know that W_∞ is also an effective Cartier divisor.

2. The fibre of $X \times \mathbb{P}^1$ on D_0 and D_∞ are isomorphic to X regardless of the base field. Thus W_0 and W_∞ can be viewed as k-dimensional subvarieties of X. Let p be the projection $X \times \mathbb{P}^1 \to X$. We know that $q = p|_W : W \to X$ is proper since it is the composition of the closed immersion $W \to X \times \mathbb{P}^1$ and the projection p. Later we will show that a proper map turns rational equivalent cycles to rational equivalent cycles. Since $\operatorname{div}(t) = [W_0]_k - [W_\infty]_k$ on W, we know that $[W_0]_k$ and $[W_\infty]_k$ are rational equivalent and thus their image in X are rational equivalent.

3. It suffices to show that for a subvariety $V \subset X$ of dimension $k+1$ and a rational function $f \in R(V)^*$. There exists W such that $[W_0]_k - [W_\infty]_k$ is exactly $\operatorname{div}(f)$ when they are viewed as k-cycles on X. We use the lemma 1.2 below to get what we want.

□

Now we can use the description above to calculate the Chow ring of affine space.

Proposition 1.3. *$A(\mathbb{A}^n_k) = \mathbb{Z} \cdot [A^n]$ if k is algebraic closed.*

In fact, it holds for any base field.

Proof. Let $Y \subset A^n$ be a properly subvariety. Choose the coordinates $z_1, \ldots, z_n$ on $\mathbb{A}^k_n$ such that the origin is not contained in Y. We define

$$W^\circ = \{(t, tz) \subset (\mathbb{A}^1 \backslash \{0\}) \times \mathbb{A}^n | z \in Y\} = V\{f(z,t) | f(z) \text{ vanishes on } Y\}$$

The fiber of W° over t is tY. Let $W \subset \mathbb{P}^1 \times \mathbb{A}^n$ be the closure of W° in $\mathbb{P}^1 \times \mathbb{A}^n$. It is also integral since W° is the image of $(\mathbb{A}^1 \backslash \{0\}) \times Y$.

The fibre of W over $t = 1$ is just Y. Since 0 is not in Y, there exists some polynomial $g(z)$ that vanishes on Y with nonzero constant item c. Thus $G(t, z) = g(z/t)$ extends to a regular function on $(\mathbb{P}^1 \backslash \{0\})_\times \mathbb{A}^n$ with constant c on the fibre $\{\infty\} \times \mathbb{A}^n$. Since $W \subset V(G)$ on $(\mathbb{P}^1 \backslash \{0\})_\times \mathbb{A}^n$, we know the fibre of W at infinite is empty. □

The proposition can be generalized to closed subschemes and coherent sheaves:

Corollary 1.1. *1. Let X be a scheme and Z be a $k+1$-dimensional subvariety of $X \times \mathbb{P}^1$. Suppose*

(a) $Z \cap \{t = 0\}$ and $Z \cap \{t = \infty\}$ are both of k-dimension.

(b) Every embedded associated point η of Z is either not a generic point of a k-dimensional subvariety, or not contained in $Z_0 \cup Z_\infty$.

Then $[Z_0]_k \sim [Z_\infty]_k$ as k-cycles on X.

2. Let X be a scheme and $\mathcal{F}$ be a coherent sheaf on X. Let i_0, i_∞ be the closed immersion identify X with $X \times D_0$ and $X \times D_\infty$. Suppose

(a) $\dim(Supp(\mathcal{F})) = k+1$ and $\dim(Supp(\mathcal{F}_0)) = \dim(Supp(\mathcal{F}_\infty)) = k$, where $\mathcal{F}_0 = i_0^* \mathcal{F}$ and $\mathcal{F}_\infty = i_\infty^* \mathcal{F}$.

(b) Every embedded associated point η of $\mathcal{F}$ is either not a generic point of a k-dimensional subvariety, or not contained in $X \times D_0 \cup X \times D_\infty$.

Then $[F_0]_k \sim [Z_\infty]_k$ as k-cycles on X.

They are direct conclusion by the functoriality of Chow rings, which we are going to discuss.

2 Functoriality

Intersection theory on a scheme is some kind of analogue of homology groups or cohomology groups on topology space. Thus it should have some functorial properties.

2.1 Proper Pushforward

At first, suppose we have a map $f : X \to Y$. The image $f(X)$ of a subvariety $Z \subset X$ is integral, but may not closed in X. If X is proper, then the image $f(X)$ is closed. Furthermore, the pushforward of a coherent sheaf along a proper morphism is still coherent. Thus we can define the proper pushfoward for cycles. Obviously, we should define the image of $[V]$ to be $[f(V)]$ topologically. But some problems occurs, we need to consider the coefficient of this map. If we just let $[V] \mapsto [f(V)]$, the map may not preserve rational equivalence.

In order to "discover" the coefficient, we assume X, Y are varieties. And $f : X \to Y$ is proper and dominant(Thus surjective). Recall we have defined the circle associated to a coherent sheaf $\mathcal{F}$ on X. Obviously, we hope that pushforward of cycles are compatible with pushforward of coherent schemes, i.e., $[f_*\mathcal{F}]$ should be $f_*[\mathcal{F}]$. With this principle, we consider $f_*\mathcal{O}_X$. Then

$$[f_*\mathcal{O}_X] = \mathrm{length}_{\mathcal{O}_{Y,\eta_Y}}(f_*\mathcal{O}_X)_{\eta_Y} = \mathrm{length}_{K(Y)}K(X) = [K(X) : K(Y)]$$

We know that the dimension of a variety is equal to the transcendental degree of its function field over the base field by Noether normalization. Since f is dominant, we know $\dim Y \leq \dim X$. Thus if the dimension of X is strictly greater than Y, the degree of $K(X)$ over $K(Y)$ should be infinite. If X and Y are of the same dimension, then $K(X)$ is algebraic and finitely generated over $K(Y)$. Hence $[K(X) : K(Y)] < \infty$.

By our discussion above, we make following definition:

Definition 2.1 (Proper Pushforward)**.** *Let $f : X \to Y$ be a proper morphism between schemes. Then for every $k \geq 0$, the map $f_* : Z_k(X) \to Z_k(Y)$ is defined by $[V] \mapsto [V : f(V)][f(V)]$ for k-dimensional subvariety $V \subset X$, where*

$$[V : f(V)] = \begin{cases} [K(V) : K(f(V))] & \dim V = \dim f(V) \\ 0 & \dim V > \dim f(V) \end{cases}$$

To be specific, let $\alpha = \sum m_i[V_i]$ be a k-cycle on X. Then

$$f_*\alpha = \sum_{W \subset Y \text{ subvariety}} \sum_{f(V_i) = W} [V_i : W]m_i[W]$$

We now check that the definition above satisfying all the properties we want:

Proposition 2.1 (Properties of proper pushforward)**.** *Let $f : X \to Y$ be a proper morphism between schemes. Then*

1. *For another proper morphism $g : Y \to Z$, we have $g_* \circ f_* = (g \circ f)_*$ as map of $Z_k(X) \to Z_k(Z)$.*
2. *Let Z be a closed subscheme of X. Then $[f_*\mathcal{O}_Z] = f_*[Z]$.*
3. *Let $\mathcal{F}$ be a coherent sheaf on X. Then $[f_*\mathcal{F}] = f_*[\mathcal{F}]$.*

Proof. 1. Let V be a subvariety of X. Then by calculation

$$\begin{aligned} g_*(f_*[V]) &= [f(V) : g(f(V))][V : f(V)][g(f(V))] \\ &= [K(f(V)) : K(g(f(V)))][K(V) : K(f(V))][g(f(V)] \\ &= [K(V) : K(g(f(V)))][g(f(V))] \\ &= [V : g(f(V))][g(f(V))] = (g \circ f)_*[V] \end{aligned}$$

2. It is a special case of the third statement.

3. Consider the diagram

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f|_z"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "i'"'] & Y \end{tikzcd}$$

where $i : Z \to X$ is closed immersion of the support of $\mathcal{F}$, Z' is the scheme-theoretical image of Z in Y. Then $\mathcal{F} = i_*\mathcal{G}$ for some coherent sheaf on Z. Thus it suffice to prove the conclusion for f closed immersion and dominant morphisms. The closed immersion case is trivial. Thus we assume f is dominant and $\mathcal{F}$ is supported on every generic point of irreducible components of X. We can also assume that $\dim X = \dim Y = k$ since $[\mathcal{F}]$ lies in $Z_k(X)$.

Suppose η is the generic point of an k-dimensional irreducible components $Z \subset Y$. Then the fibre over η only contains some generic points of irreducible components of X. Thus by generic finiteness, we can find $V \subset Y$ open containing η and $f^{-1}(V) \to V$ is finite. Thus we reduce to the case $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(R)$ such that A is finite over R. Then $\mathcal{F}$ corresponding to finite A-module M. We can also assume that η is the unique minimal prime $\mathfrak{p} \subset A$. Then the coefficient of $[f_*\mathcal{F}]$ on Z is $\mathrm{length}_{R_\mathfrak{p}}(M_\mathfrak{p})$.

Note that the coefficient of $f_*[\mathcal{F}]$ on Z is

$$\sum [\kappa(\mathfrak{q}_i) : \kappa(\mathfrak{p})] \mathrm{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$$

where $\mathfrak{q}_i$ are minimal primes of A that lying over $\mathfrak{p}$. Notice that they are the maximal ideals in $A_\mathfrak{p}$. Thus the conclusion holds by following lemma:

Lemma 2.1. *Suppose $(A, \mathfrak{m})$ is local Artinian, $(B, \mathfrak{n}_1, \dots, \mathfrak{n}_n)$ is semi-local and finite over A. Let M be a finite length A-module. Then*

$$\mathit{length}_A(M) = \sum [\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m})] \mathit{length}_{B_{\mathfrak{n}_i}}(M_{\mathfrak{n}_i})$$

□

At last, we show that f_* maps principal divisors to principal divisors. Thus f_* can be regarded as a map $A_k(X) \to A_k(Y)$.

Theorem 2.1. *Let $f : X \to Y$ be a proper morphism between schemes, $Z \subset X$ be a subvariety and $a \in K(Z)^*$. Suppose $f(Z) = W$. Then*

$$f_*[\mathit{div}_X(a)] = \begin{cases} \mathit{div}_Y(\mathit{Nm}_{K(Z)/K(W)}(a)) & \dim Z = \dim W \\ 0 & \dim Z > \dim W \end{cases}$$

Note that all the conclusions above are on cycle level. And thus we can conclude that $f_*\mathrm{div}_X(a) = 0$ in $A(Y)$. We divide the proof of the theorem into two parts.

Firstly, we show the conclusion holds if $\dim Z = \dim W$. In fact, it suffices to show that if $f : X \to Y$ is a proper dominant morphism between two varieties of the same dimension then

$$f_*\mathrm{div}_X(a) = \mathrm{div}_Y(\mathrm{Nm}_{K(Z)/K(W)}(a))$$

for $a \in K(Z)^*$. Before the argument, we explain the meaning of $\mathrm{Nm}_{E/F}(a)$. Let a be an invertible element in a field E, which is a finite extension of the field F. Then multiplying by a is a F-linear map on E. We define $\mathrm{Nm}_{E/F}(a)$ to be the determinant of such linear map induced by a. Let $b = \mathrm{Nm}_{K(Z)/K(W)}(a)$.

Now let a be an element in $K(X)$, the Weil divisor $\mathrm{div}_X(a) = \sum_Z \mathrm{ord}_Z(a)[Z]$ with sum over all prime Weil divisors Z. Then by our formula, we have

$$f_*\mathrm{div}_X(a) = \sum_Z \sum_W \mathrm{ord}_Z(a)[K(Z) : K(W)][W]$$

The sum is over all prime Weil divisors $W \subset Y$ and $Z \subset X$ such that $f(Z) = Y$. Notice that the right is a finite sum. To show that $f_*\mathrm{div}_X(a) = \mathrm{div}_Y(b)$, we only need to compare every coefficients.

The first step is to reduce the case to f finite dominant morphism between X, Y affine varieties of the same dimension. Since $[K(X) : K(Y)]$ is finite, we know there are some open $U \subset Y$ such that $f^{-1}(U) \to U$ is a finite map. But we need to calculate the coefficient for fixed W. Thus we need exert more subtle require of U by following lemma. The proof of the lemma can be seen on Stacks Project. With the lemma, we reduce the case to what we need.

Lemma 2.2. *Suppose $f : X \to Y$ is a dominant proper map between varieties of the same dimension. Suppose $y \in Y$ is the generic point of a prime Weil divisor $W \subset Y$. Then there exists an open $U \subset Y$ containing y such that $f^{-1}(U) \to U$ is finite(thus is also affine).*

The second step is to turn above into simple algebra problem. By the first step, we can translate the conclusion into an algebraic version: Let $\phi : B \to A$ be an injective(by dominance) finite map between two domains of finite type over a field. Let $\mathfrak{p} \subset B$ be a prime of height 1 and $\mathfrak{q}_1, \dots, \mathfrak{q}_m \subset A$ be all the primes of height 1 and lying over $\mathfrak{p}$. Since ϕ is finite, there are finitely many such primes. Then we need to show

$$\mathrm{ord}_{B_\mathfrak{p}}(b) = \sum_{i=1}^{m} [\kappa(\mathfrak{q}_i) : \kappa(\mathfrak{p})] \mathrm{length}_{A_{\mathfrak{q}_i}}(A_{\mathfrak{q}_i}/aA_{\mathfrak{q}_i})$$

where $a \in A$ and $b = \mathrm{Nm}_{K(B)/K(A)}(a)$. Since The general case for $a \in K(B)$ can be deduced by additivity. Thus let $A = A_\mathfrak{p}$ and $B = B_\mathfrak{p}$, we reduce the conclusion to prove that

$$\mathrm{ord}_{B}(b) = \sum_{i=1}^{m} [\kappa(\mathfrak{q}_i) : \kappa(\mathfrak{p})] \mathrm{length}_{A_{\mathfrak{q}_i}}(A_{\mathfrak{q}_i}/aA_{\mathfrak{q}_i})$$

where B is a noetherian local domain of dimension 1. And $(A, \mathfrak{q}_1, \dots, \mathfrak{q}_i)$ is a noetherian semi-local domain that is a finite extension of B. The right side is exactly $\mathrm{length}_B(A/aA)$. Then by following lemma the conclusion holds.

Lemma 2.3. *Let R be a noetherian local domain of dimension 1, V be a finite dimension k-linear space and $M \subset V$ be a finite R-module such that $M \otimes_R k = V$. Suppose $\varphi : V \to V$ is a k-linear isomorphism. Then $\varphi(M)$ is also a finite R-module such that $\varphi(M) \otimes_R k = V$. And we have*

$$\mathit{length}_R(M/M \cap \varphi(M)) - \mathit{length}_R(\varphi(M)/M \cap \varphi(M)) = \mathit{ord}_B(\mathit{det}_k(\varphi : K \to K))$$

Proof. Suppose M, M' are both R-modules satisfying the condition in the lemma, we show the left is independent of the choice of such M. Let

$$d(M, M') = \mathrm{length}_R(M/M' \cap M) - \mathrm{length}_R(M'/M \cap M')$$

It is easy to see $d(M, M') = d(M, M'') + d(M'', M')$ and $d(M, M') = -d(M', M)$. Then

$$\begin{aligned}
&\mathrm{length}_R(M/M \cap \varphi(M)) - \mathrm{length}_R(\varphi(M)/M \cap \varphi(M)) \\
&= d(M, \varphi(M)) = d(M, M') + d(M', \varphi(M)) \\
&= d(M, M') + d(M', \varphi(M')) + d(\varphi(M'), \varphi(M)) \\
&= d(M, M') + d(M', \varphi(M')) + d(M', M) = d(M', \varphi(M')) \\
&= \mathrm{length}_R(M'/M' \cap \varphi(M')) - \mathrm{length}_R(\varphi(M')/M' \cap \varphi(M'))
\end{aligned}$$

Thus we can fix a basis $e_1, \dots, e_n$ of V over k and let $M = \oplus Re_i$. Then notice that both sides are additive, i.e, we have

$$\begin{aligned}
d(M, \varphi \circ \psi(M)) = d(M, \psi(M)) + &d(\psi(M), \varphi \circ \psi(M)) = d(M, \psi(M)) + d(M, \varphi(M)) \\
\mathit{ord}_B(\det_k(\varphi \circ \psi)) = &\mathit{ord}_B(\det_k(\varphi)) + \mathit{ord}_B(\det_k(\psi))
\end{aligned}$$

Thus we only need to show the conclusion holds for generators of $\mathrm{GL}(V)$. Obviously, $\mathrm{GL}(V)$ is generated by those matrices:

1. $E_{ij}(\lambda) = \mathrm{Id}_V + \lambda E_{ij}$, where $\lambda \in R$, $1 \le i, j \le n$ be distinct integers and E_{ij} is the matrix that is 1 on the (i, j)-th element and 0 elsewhere.
2. $E_i(\lambda)$ is the diagonal matrix with $\lambda \in R$ at the i, i-th position and 1 on other elements on the diagonal.

$\square$

Now we come to the case $\dim X > \dim W$. We still reduce the case to that $f : X \to Y$ is a proper dominant morphism between two varieties. Suppose $\dim X = k$. If $\dim Y < k-1$, then the pushforward of any $k-1$-cycle is zero by definition. Thus we assume $\dim y = k-1$. Consider following commutative diagram:

$$\begin{tikzcd} Z = X \times_Y \mathrm{Spec}(K) \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \mathrm{Spec}(K) \arrow[r] & Y \end{tikzcd}$$

where $K = K(Y)$ and $\mathrm{Spec}(K) \to Y$ is the canonical map.

We claim that Z is a proper curve over K. Z is the fibre of the generic point of Y, which is proper over $\mathrm{Spec}(K)$ since f is proper. It is obviously irreducible since it contains the generic point of X. It is also reduced since $Z \hookrightarrow$ induce isomorphism on stalks. The closed points in Z correspond to the generic points of $k-1$-dimensional subvarieties of X whose image dominant Y. Thus we have a natural map $Z_{k-1}(X) \to Z_1(X)$ sending $[Z]$ to its generic point η_Z if $[Z]$ dominants Y and to 0 for other cases. And we know that $K(Z) = K(X)$. Suppose $\mathrm{div}_X(f) = \sum m_i Z_i$ for $f \in K(X)^*$. Then

$$f_* \mathrm{div}_X(f) = \sum \delta(Z_i)[K(Z_i) : K][Y]$$

where $\delta(Z_i)$ is 1 if Z_i dominant Y and 0 if not. Also we know that $\mathrm{div}_Z(f) = \sum \delta(Z_i) m_i [\eta_{Z_i}]$ and we have

$$f'_* \mathrm{div}_Z(f) = \sum \delta(Z_i)[\kappa(\eta_{Z_i}) : K][\mathrm{Spec}(K)] = \sum \delta(Z_i)[K(Z_i) : K][\mathrm{Spec}(K)]$$

Hence $f_* \mathrm{div}_Y(f) = 0$ if and only if $f'_* \mathrm{div}_Z(f) = 0$. By the lemma below, we draw the conclusion:

Lemma 2.4. *Let K be a field and X be an integral 1-dimensional scheme with proper structure map $c : X \to \mathrm{Spec}(K)$. Then*

$$\sum\nolimits_{x \in X \ \mathit{closed}} [\kappa(x) : K]\, \mathit{ord}_{\mathcal{O}_{X,x}}(f) = 0$$

for every $f \in K(X)^*$.

Proof. Let Y be the graph of f as we defined in 1.2. Conside the diagram

$$\begin{tikzcd} Y \arrow[r, "p"'] \arrow[d, "q"'] & X \arrow[d, "c"] \\ \mathbf{P}^1_K \arrow[r, "c'"] & \mathrm{Spec}(K) \end{tikzcd}$$

where p and q are closed immersion composited with projections and c, c' are structure maps. We know that $c_* \mathrm{div}_X(f) = c'_* p_* \mathrm{div}_Y(f)$. The image of Y in $\mathbb{P}^1_K$ is closed and irreducible. If $p(Y)$ is a single closed point then $p_* \mathrm{div}_Y(f) = 0$. Otherwise $p : Y \to \mathbb{P}^1_K$ is dominant. We claim that p is finite and $p_* \mathcal{O}_Y$ is a locally free module of rank d. Then by the lemma 2.6 and the fact $\mathrm{div}_Y(f) = [q^{-1}D_0] - [q^{-1}D_\infty]$, we know that

$$c'_* q_* \mathrm{div}_Y(f) = c'_*(d[D_0] - d[D_\infty]) = 0$$

It remains to show that p is finite and $p_* \mathcal{O}_Y$ is a locally free module of rank d. By lemma 2.2, we know p is finite. $p_* \mathcal{O}_Y$ is obviously torsion free. Since the local ring of every closed point of $\mathbb{P}^1_K$ is discrete valuation ring, we know $p_* \mathcal{O}_Y$ is free thus of constant finite rank $d = [K(Y) : K(\mathbb{P}^1_K)]$. □

At last, we define the degree of zero-cycles on complete scheme:

Definition 2.2. *Suppose X is a complete scheme with base field K. Then the degree of $\alpha = \sum_P n_P[P]$ is*

$$\deg(\alpha) = \int_X \alpha = \sum_P n_P [\kappa(P) : K]$$

By our argument above, we know that deg is an additive function from $A_0(X) \to \mathbb{Z}$. Suppose we have a morphism $f : X \to Y$ between complete schemes. Then $\int_X \alpha = \int_Y f_*\alpha$ for $\alpha \in A_0(X)$. Sometimes we extend deg to $A_*(X)$ by setting $\deg(\beta) = 0$ if $\beta \notin A_0(X)$.

An interesting example is the normalization of varieties. Suppose X is a variety and $\nu : \tilde{\to} X$ is its normalization. Thus $\tilde{X}$ is regular in codimension one. Then $\mathrm{ord}_Z(g)$ is the valuation for $g \in K(\tilde{X})$ and $Z \subset \tilde{X}$ a prime divisor. Notice that the function field on $\tilde{X}$ is the same as the function field on X. Thus for $g \in K(X)^*$, we have

$$\mathrm{ord}_V(g) = \sum \mathrm{ord}_{\tilde{V}}(g)[K(\tilde{V}) : K(V)]$$

for all $V \subset X$ prime divisor and $\tilde{V} \subset \tilde{X}$ dominant V. We can use this definition to define the principal divisors for general schemes.

2.2 Flat Pullback

Now we define the pullback for flat morphism of some fixed relative dimension. A morphism $f : X \to Y$ is flat of relative dimension r means for every point(may not be closed) $y \in Y$, the fibre X_y is either empty or of pure dimension r. We can show that if f is flat of relative dimension r and $Z \subset Y$ is an integral closed subscheme then $f^{-1}(Z)$ is pure of dimension $r + \dim Z$. And $g \circ f$ is flat of relative dimension $r' + r$ if $g : Y \to W$ is flat of relative dimension r'. For details of discussion on such morphisms, see Stacks Project.

Suppose we have such a map $f : X \to Y$. Then the inverse image of any k-dimensional subvariety Z of Y is pure of dimension $k + r$. And every irreducible component of $f^{-1}[Z]$ dominants Y by the restriction of f. As the same way in the proper pushforwards, we hope to have $f^*[\mathcal{F}] = [f^*\mathcal{F}]$. Thus we take $f^*[Z] = [f^*\mathcal{O}_Z]$. The right side is exactly the cycle associated to the scheme theoretical inverse $f^{-1}(Z)$.

Definition 2.3 (Flat pullback)**.** *Let $f : X \to Y$ be a flat morphism of relative dimension r between schemes. Then $f^* : Z_k(X) \to Z_{k+r}(Y)$ is defined by $f^*[V] = [f^{-1}(V)]$ for subvarieties $V \subset Y$.*

Now we need to check that our definition preserves rational equivalence and compatible with functoriality. Since the base change of f is still flat and of relative dimension r, we can assume Y is integral to check that flat pullback preserves rational equivalence. Let $X_1, \dots, X_\ell$ be irreducible components of X, which are all of dimension $r + \dim Y$. Then $f_i = f|_{X_i} : X_i \to Y$ is a dominant morphism between varieties. Thus $g \in K(Y)$ can also be regarded as an element in $K(X_i)$. Suppose X_i is of geometric multiplicity n_i. Then we have following proposition:

Proposition 2.2. *Let X be a scheme and Y be a variety. Suppose $f : X \to Y$ is a flat morphism relative dimension r. Let X_i be the irreducible components of X with geometric multiplicity n_i for $i = 1, \dots, \ell$. Then we have*

$$f^* div_Y(g) = \sum n_j div_{X_i}(g)$$

for $g \in K(Y)$.

Let $Z \subset X$ be a $\dim X - 1$ dimensional subvariety. Then the closure of $f(Z)$ is either Y or a $\dim Y - 1$ dimensional subvariety Z'. If $Z \to Y$ is dominant, then the coefficient of $[Z]$ are equal to 0 on both sides. Thus we assume that $\overline{f(Z)} = Z'$. By additivity, we can assume that g is regular in $\xi_{Z'}$. Let $A = \mathcal{O}_{Y,Z'}$ and $B = \mathcal{O}_{X,Z}$. Then B is flat over A and the coefficient of Z in the right side is

$$\begin{aligned}
&\mathrm{length}_A(A/g) \cdot \mathrm{length}_B((f^*\mathcal{O}_Z)_{\xi_Z}) \\
&= \mathrm{length}_A(A/g) \cdot \mathrm{length}_B(B \otimes \kappa(A)) \\
&= \mathrm{length}_B(B/gB)
\end{aligned}$$

The last equation holds by following argument: Choose

$$0 \subset M_0 \subset M_1 \subset \ldots \subset M_e = A/g$$

such that $M_i/M_{i-1} \simeq \kappa(A)$. Then $\mathrm{length}_A(A/g) = e$. Thus consider

$$0 \subset M_0 \otimes B \subset M_1 \otimes B \subset \ldots \subset M_e \otimes b = B/g$$

And we know that $M_i \otimes B / M_{i-1} \otimes B \simeq \kappa(A) \otimes B$.

Then we consider the left side. B is of dimension 1 and its minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_c$ correspond to those irreducible components of B that contain Z. Thus the left side is

$$\sum_{i=1}^{c} \mathrm{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i}) \mathrm{ord}_{B/\mathfrak{q}_i}(g)$$

By following proposition, the theorem holds:

Lemma 2.5. *Let R be a local noetherian ring and M be a finite R-module. Suppose $x \in R$, $\dim(Supp(M)) \leq 1$ and $\dim(Supp(M/xM)) \leq 0$. Suppose $Supp(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$. Then*

$$length_R(M/xM) - length_R({}_xM) = \sum_{i=1}^{t} ord_{R/\mathfrak{q}_i}(x) length_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$$

Proof. Briefly, we take following four steps:

The first step, suppose we have a exact sequence $0 \to M' \to M \to M'' \to 0$. If the conclusion holds for two of the modules then it also holds for the third.

The second step, if M only supports on $\mathfrak{m}$. Then M is a finite dimensional $\kappa(R)$-vector space and both sides equals to zero. Thus we consider $M \to \bigoplus_{i=1}^{t} M_{\mathfrak{q}_i}$, whose kernel and cokernel both supports on $\mathfrak{m}$. We reduce the case to $\mathrm{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$.

The third step, we know that $\mathfrak{q}R_{\mathfrak{q}}$ is nilpotent. Thus consider following sequence:

$$0 = \mathfrak{q}^m M \subset \mathfrak{q}^{m-1} M \ldots \subset \mathfrak{q}M \subset M$$

We reduce the case to M being $R/\mathfrak{q}$-module.

The last step. We assume that R is a noetherian local domain of dimension one and $K = Q(R)$. The torsion part of M support on $\mathfrak{m}$. Thus we can assume that M is torsion-free, i.e., $M \hookrightarrow M \otimes K = V$. Then $x : M \to M$ is injective and the left side is $\mathrm{length}_R(M/xM)$. By lemma 2.3, we know that it is equal to

$$\mathrm{ord}(\det(x : V \to V)) = \mathrm{ord}(x^{\dim_K V}) = \dim_K V \cdot \mathrm{ord}_R(x)$$

□

As a corollary of the lemma, we get:

Corollary 2.1. *Let X be a scheme of pure dimension n, with irreducible components $X_1, \ldots, X_r$ and geometric multiplicities $m_1, \ldots, m_r$. Let D be an effective Cartier divisor on X. Then*

$$[D] = \sum_{i=1}^{r} m_i [D_i]$$

where $D_i = D \cap X_i$.

Proof. It is suffices to calculate the coefficient on every irreducible $n-1$-dimensional subvariety Z. Since D is Cartier, we assume that there exists a regular element $g \in A = \mathcal{O}_{X,Z}$ such that D is determined by g locally. Then the left side is $\mathrm{ord}_A(g)$ and the right side is

$$\sum_{i=1}^{r} m_i \mathrm{ord}_{A/(\mathfrak{q}_i)}(g)$$

where $\mathfrak{q}_i$ is the prime of A corresponding to X_i. The conclusion holds by the lemma 2.5 □

Then we check that the flat pullback is compatible with pullback of coherent sheaves.

Proposition 2.3. *Let $f : X \to Y$ be a flat morphism of relative dimension r between schemes.*

1. *Let $Z \subset Y$ be a closed subscheme of dimension k. Then $[f^{-1}(Z)] = f^*[Z]$;*

2. Let $\mathcal{F}$ be a coherent sheaf on Y. Suppose $\dim(Supp(\mathcal{F})) = k$. Then $f^*[\mathcal{F}] = [f^*\mathcal{F}]$ in $Z_{k+r}(X)$.

Proof. We only need to prove the second statement. Let $W \subset \mathrm{Supp}(\mathcal{F})$ be a subvariety of dimension k and ξ be its generic point. Let $W' \subset X$ be a $k+r$ subvariety mapping into W with generic point ξ'. Suppose $A = \mathcal{O}_{Y,\xi}$, $B = \mathcal{O}_{X,\xi'}$ and $M = \mathcal{F}_{\xi'}$. Then $(f^*\mathcal{F})_{\eta'} = B \otimes_A M$. Then the left side is $\mathrm{length}_A(M) \cdot \mathrm{length}_B(B \times \kappa(A))$ while the right side is $\mathrm{length}_B(B \otimes_A M)$. Thus our conclusion holds. □

At last, we may ask the relation of proper pushforward and flat pullback.

Proposition 2.4 (Cohomology with base change)**.** *Consider the fibre product*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

where g is flat and f is proper. Then for all $\alpha \in Z_*(X)$, we have

$$f'_*(g')^*\alpha = g^* f_*\alpha$$

Proof. Let $V \subset X$ be a subvariety. Then by cohomology with base change we have natural isomorphism

$$Rf'_*(g')^*\mathcal{O}_V \simeq g^* Rf_*\mathcal{O}_V$$

Taking the first cohomology sheaves we know the proposition holds. □

Another important case is the finite flat morphism.

Lemma 2.6. *Suppose $f : X' \to X$ be a finite and flat morphism. Then $f_*f^* : Z_*(X) \to Z_*(X)$ is multiplication by d, where $d = [K(X') : K(X)]$.*

Proof. Let V be a subvariety of X. By base change, we can assume that $V = X$ and V is affine. Then

$$f_*f^*[V] = [f_*f^*\mathcal{O}_V] = [\mathcal{O}_V^{\oplus d}] = d[V]$$

□

At last, we make a remark. The pullback can also be defined for morphisms between smooth quasi-projective varieties without the restriction of flatness.

2.3 MV-Sequence

The most elementary case of proper morphisms is closed immersion and the counterpart of flat morphisms is open immersion. Let X be a scheme and $U \subset X$ be an open. Let $Y = X - U$ with the reduced structure.(In fact, we always have $Z_k(Y) = Z_k(Y_{red})$.) Then we obviously have the exact sequence

$$0 \to Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \to 0$$

where $i : Y \to Z$ and $j : U \to Z$ are given immersions. We may guess that the exactness can pass to Chow groups. In fact, $A_k(Y) \to A_k(X)$ cannot be injective in general. For example, if Y is a variety then Y itself may be rational equivalent to zero circle in a larger space. However, the sequence above is always right exact.

To prove the fact, recall that $A_k(X)$ has a representation

$$Z_{k+1}(X \times \mathbb{P}^1) \to Z_k(X) \to A_k(X)$$

where the first map takes $[V]$ to $V_0 - V_\infty$ if V is not contained in $X \times D_0$ or $X \times D_\infty$ and to 0 for other cases.

Proposition 2.5 (MV-sequence)**.** *Let X be a scheme.*

1. *Suppose $U \subset X$ is open and $Y = X - U$. Then we have exact sequence*

$$A_k(Y) \to A_k(X) \to A_k(U) \to 0$$

2. *Suppose X_1, X_2 are closed subschemes of X, then there are exact sequences*

$$A_k(X_1 \cap X_2) \to A_k(X_1) \oplus A_k(X_2) \to A_k(X_1 \cup X_2) \to 0$$

Proof. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & Z_{k+1}(Y \times \mathbb{P}^1) \arrow[r] \arrow[d] & Z_{k+1}(X \times \mathbb{P}^1) \arrow[r] \arrow[d] & Z_{k+1}(U \times \mathbb{P}^1)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_k(Y) \arrow[r] & Z_k(X) \arrow[r] & Z_k(U) \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma our conclusion holds. The argument for the second sequence is similar. □

In fact, we have a generalized version for the second one:

Lemma 2.7. *Consider the fibre product*

$$\begin{tikzcd}
Y' \arrow[r, "j"] \arrow[d, "q"'] & X' \arrow[d, "p"] \\
Y \arrow[r, "i"'] & X
\end{tikzcd}$$

where i is a closed embedding, p proper and p induce isomorphism $X' - Y'$ to $X - Y$. Then the sequence

$$A_k(Y') \to A_k(Y) \oplus A_k(X') \to A_k(X) \to 0$$

is exact.

Corollary 2.2. *A direct conclusion is that $A(U) = \mathbb{Z}[U]$ for every nonempty open set U of $\mathbb{A}^n$.*

The MV-sequence gives us the method to calculate the Chow ring of some varieties. Recall that given a CW-complex in topology, its k-th cohomology group is generated by the class of its k-cells. We have such an analog in algebraic geometry.

Definition 2.4. *A stratification on a scheme X is a finite collection of disjoint irreducible and locally closed subschemes $U_1, \dots, U_n$ of X such that*

1. *$\overline{U_i}$ is a union of some U_j.*
2. *$\bigcup U_i = X$.*

We usually call that U_i the strata of the stratification and $Y_i = \overline{U_i}$ the closed strata.

We say that a stratification is affine(resp. quasi-affine) if each U_i is isomorphic to some $\mathbb{A}^k$(resp. open subset of $\mathbb{A}^k$). For example, the complete flag of subspaces $\mathbb{P}_k^0 \subset \mathbb{P}_k^1 \subset \ldots \subset \mathbb{P}_k^n$ gives an affine stratification of $\mathbb{P}_k^n$ with strata $U_i = \mathbb{P}_k^i \backslash \mathbb{P}^{i-1} \simeq \mathbb{A}^i$.

Proposition 2.6. *If a scheme X has a quasi-affine stratification, then $A(X)$ is generated by the classes of the closed strata.*

Proof. We need to use induction on the number of strata. If U_0 is a minimal stratum, then $A(U_0) = \mathbb{Z} \cdot [U_0]$ as we known. A minimal stratum is always closed. Thus consider the exact sequence

$$A(U_0) \to A(X) \to A(X \backslash U_0) \to 0$$

By induction we win. □

In fact, we have more strong theorem for affine stratification:

Theorem 2.2 (Torato(2014))**.** *The classes of the strata in an affine stratification of a scheme X form a free basis of $A(X)$.*

3 Projective space

The most elementary example is the projective space.

Theorem 3.1. *The Chow ring of $\mathbb{P}^n$ is $A(\mathbb{P}^n) = \mathbb{Z}[\xi]/(\xi^{n+1})$, where $\xi \in \mathbb{A}^1(\mathbb{P}^n)$ is the rational equivalence class of a hyperplane. The class of a subvariety of codimension k and degree d is $d\xi^k$.*

Proof. By lemma 2.2, we know that $A_k(\mathbb{P}^n)$ is freely generated by $\overline{\mathbb{A}^k} = \mathbb{P}^k$, thus by the class of any k-dimensional plane $L \subset \mathbb{P}^n$. We know that in general the intersection of k-dimensional and k'-dimensional planes is a $k + k' - n$-dimensional plane. Thus our conclusion holds. $\square$

Corollary 3.1. *A morphism from $\mathbb{P}^n$ to a quasi-projective variety of dimension strictly less than n is constant.*

Proof. Let $\mathbb{P}^n_k \to X \subset \mathbb{P}^m_k$ be the map. Without loss of generality, we assume the map is surjective onto X. Then all the hyperplanes that not meet the associated point of X restricts to an effective Cartier divisor D on $\mathbb{P}^n_k$. Fix such a hyperplane H. Consider any closed point x that not contained in such a hyperplane. Suppose the fibre of x is 0-dimensional thus finite. Then by Zariski main theorem we know that the fibre across the generic point of $\mathbb{P}^n_k$ is quasi-finite there, which contradict the fact that $\dim X < n$. Thus the fibre of x is of dimension more than 0. Thus it must meet D, which contradict our assumption. $\square$